Existence result for a class of functional integral equations via the measure of non-compactness and applications

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Abstract. In the present paper, by expedient assumptions and using the meaning of measure of non-compactness and essential fixed point theorems such as Darbo's theorem, we present an existence of solutions for some nonlinear functional integral equations on $C[0,T]$. Our existence results subtend many key integral and functional equations that emerge in nonlinear analysis and its applications. We show applications of the obtained results for specific scenarios of known equations.

Keywords: nonlinear functional integral equations, measure of non-compactness, fixed point theorem, Darbo condition, Banach algebra

1. Introduction

Some problems investigated in the vehicular traffic theory, queuing theory, biology and etc., lead to functional integral equations, see [7, 13]. For instance, the most important frequently considered integral equations, in linear or its nonlinear counterparts cases, are the Hammerstein integral equation and its generalization the Urysohn integral equation.

In this study, we discuss on some of these equations in the following general form:

$$\varphi(t) = \psi(t, \varphi(t)) + \int_0^t \rho(t, \tau, \varphi(\tau))d\tau, \tag{1}$$

where $t \in [0, T]$. Many important integral equations, for example; the nonlinear Volterra integral equations such as the Urysohn type integral equation, which were considered in many papers and monographs [7, 11, 13, 14], are especial cases of Eq. (1). The main tool for study on existence of solutions for functional integral equations (1) is the fixed point theorem which satisfies the Darbo condition [1, 2, 8, 9], with respect to a measure of non-compactness in the Banach algebra of continuous functions on the interval $[0, T]$. Fixed point theorems have many important applications in nonlinear analysis literature [1, 9, 12, 10, 15].

The rest of this paper is organized as follows. Section 2 is devoted to collect some definitions and auxiliary results, which will be used in this work. In Section 3, applying the technique associated with measure of non-compactness, we prove an existence theorem for Eq. (1). In the last section, we present some examples that verify the application of our reviewed results for nonlinear functional integral equations.

2. Notation and auxiliary facts

We recall some basic definitions and properties of measure of non-compactness [2, 3, 5, 6, 8], which are utilized for obtaining our main results.

Definition 1. *Let (X, d) be a complete metric space and B the family of nonempty and bounded subsets of X. A function ξ, defined on B in $\mathbb{R}_+ = [0, \infty)$, is called the measure of non-compactness on X if $\xi(\overline{co}(\mathcal{B})) = \xi(\mathcal{B})$ for all bounded subsets $\mathcal{B} \in B$, where $\overline{co}(\mathcal{B})$ denotes the convex closure of $\mathcal{B}$.*

For $\mathcal{B}, \mathcal{B}_1, \mathcal{B}_2 \in B$, the measure of non-compactness ξ is said to be:

1. Monotone: if $\mathcal{B}_1 \subseteq \mathcal{B}_2$ implies $\xi(\mathcal{B}_1) \subseteq \xi(\mathcal{B}_2)$;
2. Regular: if $\xi(\mathcal{B}) = 0$ is equivalent to the relative compactness of $\mathcal{B}$ in X;
3. Nonsingular: if $\xi(\mathcal{B}) = 0$ for every finite set $\mathcal{B}$;
4. Semi-additive: if $\xi(\mathcal{B}_1 \cup \mathcal{B}_2) = \max\{\xi(\mathcal{B}_1), \xi(\mathcal{B}_2)\}$.

As an example of a measure of non-compactness, possessing all above properties, we may consider the non-compactness measure of Hausdorff ξ_1 defined on $\mathcal{B} \in B$ as follows,

$$\xi_1(\mathcal{B}) = \inf\{\varepsilon > 0 : \mathcal{B} \text{ can be covered by a finite number of balls with radii} \leq \varepsilon\}.$$

Without confusion, the Kuratowski measure of non-compactness ξ_2 defined by

$$\xi_2(\mathcal{B}) = \inf\{\varepsilon > 0 : \mathcal{B} \text{ can be covered by finitely many sets with diameter} \leq \varepsilon\},$$

where the diameter of $\mathcal{B}_i$ is defined by $diam\mathcal{B}_i = \sup\{|b_1 - b_2| : b_1, b_2 \in \mathcal{B}_i\}, i = 1, 2, \ldots, m$.
It is well-known that the Kuratowski measure of non-compactness ξ_2, verifies properties 1-4, as well as the Hausdorff measure of non-compactness.

If X is a Banach space, then for the Kuratowski or Hausdorff measure of non-compactness ξ, we also have:

5. Semi-homogeneity: $\xi(\alpha\mathcal{B}) = |\alpha|\xi(\mathcal{B})$, where $\alpha \in \mathbb{R}$ and $\alpha\mathcal{B} = \{\alpha b : b \in \mathcal{B}\}$;
6. Algebraic semi-additivity: $\xi(\mathcal{B}_1 + \mathcal{B}_2) \leq \xi(\mathcal{B}_1) + \xi(\mathcal{B}_2)$, where $\mathcal{B}_1 + \mathcal{B}_2 = \{b_1 + b_2 : b_1 \in \mathcal{B}_1, b_2 \in \mathcal{B}_2\}$;

7. Lipschitzianity: $|\xi(\mathcal{B}_1) - \xi(\mathcal{B}_2)| \leq L_\xi d_l(\mathcal{B}_1, \mathcal{B}_2)$, where $L_{\xi_1} = 1, L_{\xi_2} = 2$ and $d_l(\mathcal{B}_1, \mathcal{B}_2)$ denotes the Hausdorff metric of $\mathcal{B}_1$ and $\mathcal{B}_2$, i.e.

$$d_l(\mathcal{B}_1, \mathcal{B}_2) = \max\{\sup_{t \in \mathcal{B}_2} d(t, \mathcal{B}_1),\ \sup_{t \in \mathcal{B}_1} d(t, \mathcal{B}_2)\},$$

here $d(.,.)$ is the distance from an element of X to a set of X;

8. Continuity: for every $\mathcal{B} \in B$ and for all $\varepsilon > 0$, there is $\delta > 0$ such that $|\xi(\mathcal{B}) - \xi(\mathcal{B}_1)| < \varepsilon$ for all $\mathcal{B}_1$ satisfying $d_l(\mathcal{B}, \mathcal{B}_1) < \delta$.

Theorem 1. *The relationship of the Hausdorff measure of non-compactness* ξ_1, *and the Kuratowski measure of non-compactness* ξ_2, *on* $\mathcal{B} \in B$ *is*

$$\xi_1(\mathcal{B}) \leq \xi_2(\mathcal{B}) \leq 2\xi_1(\mathcal{B}).$$

The expressed property allows us to characterize solutions of Eq. (1), and will be used in the next section. Further facts concerning measures of non-compactness and their properties may be found in [2, 5].

Theorem 2 (Darbo fixed point theorem) ([2, 9])**.** *Let* $D \neq \emptyset$ *be a bounded, closed and convex subset of a Banach space* X, ξ_2 *be the Kuratowski measure of non-compactness on* X *and suppose that* $\Lambda : D \to D$ *is a continuous operator such that there exists a constant* $\eta \in [0, 1)$ *with* $\xi_2(\Lambda\Phi) \leq \eta\xi_2(\Phi)$ *for all* $\Phi \in D$. *Then* Λ *has a fixed point in* D.

In what follows, we will work in the classical Banach space $C[0, T]$, consisting of the all real and continuous functions defined on the interval $[0, T]$. This space is equipped with the standard(uniform) norm

$$\|x\| = \sup\{|x(t)| :\ t \in [0, T]\}.$$

Obviously, the space $C[0, T]$ also has the structure of a Banach algebra.

Now, we present the definition of a special measure of non-compactness in $C[0, T]$ which will be used in the sequel, a measure that was introduced and studied in [4, 11]. To do this, let us fix a subset Φ belong to the family of all nonempty and bounded subsets of $C[0, T]$. For $\varepsilon > 0$ and $\varphi \in \Phi$ denote by $\mu(\varphi, \varepsilon)$ the *modulus of continuity* of φ defined by

$$\mu(\varphi, \varepsilon) = \sup\{|\varphi(t_1) - \varphi(t_2)| :\ |t_1 - t_2| \leq \varepsilon,\ t_1, t_2 \in [0, T]\}.$$

Further, we set

$$\mu(\Phi, \varepsilon) = \sup\{\mu(\varphi, \varepsilon) : \varphi \in \Phi\},$$

$$\mu_0(\Phi) = \lim_{\varepsilon \longrightarrow 0} \mu(\Phi, \varepsilon),$$

the function $\mu_0(\Phi)$ is a regular measure of non-compactness in the space $C[0, T]$.

3. Main result

According to the announcement given in the Introduction, we study the solvability of the nonlinear functional integral equations (1) for $\varphi \in C[0,T]$. The assumptions are formulated for Eq. (1), namely, we assume the following hypotheses.

(H_1) $\psi : [0,T]\times\mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there exist nonnegative constants β such that $|\psi(t,0)| \leq \beta$, for $t \in [0,T]$. Also, exists the continuous function $\psi_1 : [0,T] \longrightarrow [0,T]$ such that

$$|\psi(t,\varphi_1) - \psi(t,\varphi_2)| \leq \psi_1(t)|\varphi_1 - \varphi_2|,$$

and let $\kappa = \max\{|\psi_1(t)| : t \in [0,T]\}$.

(H_2) $\rho(t,\tau,\varphi)$: $[0,T] \times [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and satisfies in sublinear condition, i.e.

$$|\rho(t,\tau,\varphi)| \leq \gamma + \delta|\varphi|,$$

for $\gamma, \delta \in \mathbb{R}_+$, $t, \tau \in [0,T]$ and $\varphi \in \mathbb{R}$.

(H_3) $\kappa < 1 - \delta T$.

Theorem 3. *Under the tacit assumptions* $(H_1) - (H_3)$ *above, Eq.* (1) *has at least one solution in the Banach algebra* $C = C[0,T]$.

Proof. To prove this result, we need to define operators Λ on the space C[0, T] in the following way

$$(\Lambda\varphi)(t) = \psi(t,\varphi(t)) + \int_0^t \rho(t,\tau,\varphi(\tau))d\tau.$$

So, Λ transforms the Banach algebra $C[0,T]$ into itself. Let us fix $\varphi \in C[0,T]$, then using our assumptions for $t \in [0,T]$, we get

$$\begin{aligned}
|(\Lambda\varphi)(t)| &= \left|\psi(t,\varphi(t)) + \int_0^t \rho(t,\tau,\varphi(\tau))d\tau\right| \\
&\leq |\psi(t,\varphi(t)) - \psi(t,0)| + |\psi(t,0)| + \left|\int_0^t \rho(t,\tau,\varphi(\tau))d\tau\right| \\
&\leq \kappa|\varphi(t)| + \beta + \left|\int_0^t \rho(t,\tau,\varphi(\tau))d\tau\right| \\
&\leq \kappa|\varphi(t)| + \beta + T(\gamma + \delta|\varphi(t)|) \\
&\leq (\kappa + \delta T)|\varphi| + (\beta + \gamma T).
\end{aligned}$$

Obviously, in view of the assumptions $(H_1) - (H_3)$, we have $\beta + \gamma T < \infty$ and $\kappa + \delta T < 1$, also the operator Λ transforms $\mathcal{B}_r$ into it self for $r = \frac{\beta+\gamma T}{1-(\kappa+\delta T)}$. Where $\mathcal{B}_r(\theta)$ denotes the closed ball with radius r centered at θ as the zero element of $C[0,T]$.

Next, we show that the operator Λ is continuous on the ball $\mathcal{B}_r(\theta)$. To do this, fix $\varepsilon > 0$ and take arbitrary $\varphi_1, \varphi_2 \in \mathcal{B}_r(\theta)$ such that $\|\varphi_1 - \varphi_2\| \leq \varepsilon$. Then for $t \in [0, T]$, we get

$$
\begin{aligned}
|(\Lambda\varphi_1)(t) - (\Lambda\varphi_2)(t)| &= \Big|\psi(t, \varphi_1(t)) + \int_0^t \rho(t, \tau, \varphi_1(\tau))d\tau - \psi(t, \varphi_2(t)) \\
&\quad - \int_0^t \rho(t, \tau, \varphi_2(\tau))d\tau\Big| \\
&\leq \kappa|\varphi_1 - \varphi_2| + \int_0^t |\rho(t, \tau, \varphi_1(\tau)) - \rho(t, \tau, \varphi_2(\tau))|d\tau \\
&\leq \kappa\varepsilon + T \cdot \mu(\rho, \varepsilon),
\end{aligned}
$$

where

$$
\mu(\rho, \varepsilon) = \sup\{|\rho(t, \tau, \varphi_1) - \rho(t, \tau, \varphi_2)| : |\varphi_1 - \varphi_2| \leq \varepsilon, \varphi_1, \varphi_2 \in [-r, r], t, \tau \in [0, T]\}.
$$

On the other hand, using the uniformly continuous of the function $\rho = \rho(t, \tau, \varphi)$ on the bounded subset $[0, T] \times [0, T] \times [-r, r]$, we infer that $\mu(\rho, \varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Thus, the above assessment shows that the operator Λ is continuous on $\mathcal{B}_r(\theta)$.

Now, we prove that the operator Λ satisfies the Darbo condition with respect to the measure μ_0, defined in Section 2, on the ball $\mathcal{B}_r(\theta)$. Take a nonempty subset Φ of $\mathcal{B}_r(\theta)$ and $\varphi \in \Phi$, then for a fixed $\varepsilon > 0$ and $t_1, t_2 \in [0, T]$ such that, without loss of generality, we may assume that $t_1 \leq t_2$ and $t_2 - t_1 \leq \varepsilon$, we obtain

$$
\begin{aligned}
&|(\Lambda\varphi)(t_2) - (\Lambda\varphi)(t_1)| \\
&\quad = \Big|\psi(t_2, \varphi(t_2)) + \int_0^{t_2} \rho(t_2, \tau, \varphi(\tau))d\tau - \psi(t_1, \varphi(t_1)) \\
&\qquad - \int_0^{t_1} \rho(t_1, \tau, \varphi(\tau))d\tau\Big| \\
&\quad \leq |\psi(t_2, \varphi(t_2)) - \psi(t_2, \varphi(t_1))| + |\psi(t_2, \varphi(t_1)) - \psi(t_1, \varphi(t_1))| \\
&\qquad + \left|\int_0^{t_2} \rho(t_2, \tau, \varphi(\tau))d\tau - \int_0^{t_1} \rho(t_1, \tau, \varphi(\tau))d\tau\right| \\
&\quad \leq \kappa|\varphi(t_2) - \varphi(t_1)| + \mu_\psi(\varepsilon, .) \\
&\qquad + \left|\int_0^{t_1} \Big(\rho(t_2, \tau, \varphi(\tau)) - \rho(t_1, \tau, \varphi(\tau))\Big)d\tau + \int_{t_1}^{t_2} \rho(t_2, \tau, \varphi(\tau))d\tau\right| \\
&\quad \leq \kappa|\varphi(t_2) - \varphi(t_1)| + \mu_\psi(\varepsilon, .) + T\mu_\rho(\varepsilon, ., .) + M\varepsilon. \qquad (2)
\end{aligned}
$$

For simplicity, we employ of the following notations

$$
\mu_\psi(\varepsilon, .) = \sup\left\{|\psi(t, \varphi) - \psi(\hat{t}, \varphi)| : \ \varphi \in [-r, r], \ |t - \hat{t}| \leq \varepsilon, \ t, \hat{t} \in [0, T]\right\},
$$

$$
\mu_\rho(\varepsilon, ., .) = \sup\left\{|\rho(t, \tau, \varphi) - \rho(\hat{t}, \tau, \varphi)| : \ \varphi \in [-r, r], \ |t - \hat{t}| \leq \varepsilon, \ t, \hat{t}, \tau \in [0, T]\right\},
$$

$$
M = \sup\left\{|\rho(t, \tau, \varphi)| : \ \varphi \in [-r, r], \ t, \tau \in [0, T]\right\}.
$$

Then, using relation (2), yields

$$\mu(\Lambda\varphi,\varepsilon) \leq \kappa\mu(\varphi,\varepsilon) + \mu_\psi(\varepsilon,.) + T\mu_\rho(\varepsilon,.,.) + M\varepsilon.$$

As in because our assumptions, we conclude that the functions $\psi = \psi(t,\varphi)$, $\rho = \rho(t,\tau,\varphi)$ are uniformly continuous on the sets $[0,T]\times\mathbb{R}$ and $[0,T]\times[0,T]\times\mathbb{R}$, respectively. Hence, we figure out $\mu_\psi(\varepsilon,.) \longrightarrow 0$, and $\mu_\rho(\varepsilon,.,.) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Consequently, we get

$$\mu_0(\Lambda\Phi) \leq \kappa\mu_0(\Phi). \tag{3}$$

Finally, we understand the operator Λ satisfies the Darbo condition on $\mathcal{B}_r(\theta)$ respect to the measure μ_0 with constant κ. Also, under the assumption (H_3), we know that $\kappa < 1$ and operator Λ is a contraction on $\mathcal{B}_r(\theta)$ with respect to μ_0. Therefore, applying Theorem 2 we conclude that Λ has at least one fixed point in $\mathcal{B}_r(\theta)$. Consequently, functional integral equations (1) has at least one solution in $\mathcal{B}_r(\theta)$. This completes the proof. $\square$

4. Applications

In this section, we present some examples of the classical integral and functional equations considered in nonlinear analysis, which are particular cases of Eq. (1) and consequently, the existence of their solutions can be established using Theorem 3.

Example 1. Setting $\psi(t,s) = f(t)$, Eq. (1) reduces to the well-known nonlinear Volterra-Urysohn integral equation

$$\varphi(t) = f(t) + \int_0^t \rho(t,\tau,\varphi(\tau))d\tau. \tag{4}$$

The equation of this type appears in many applications. For example, it can be applied in modeling the some problems in engineering, economics and physics. Also, a lot of problems considered in the theory of partial differential equations lead us to the Urysohn integral equation.

Example 2. In Eq. (4) if we set $\rho(t,\tau,\varphi(\tau)) = k(t,\tau)v(\tau,\varphi(\tau))$, a nonlinear integral equation will be appeared known as the Hammerstein type. Solution of many problems in mathematical physics, control theory, the dynamic model of chemical reactor, and studying of an elliptic partial differential equation with nonlinear boundary conditions, encounter with this equation type.

Example 3. This example shows that in general the assumptions of our existence Theorem 3, can be easily verified in objective situations. For the following functional integral equations

$$\begin{aligned}
\varphi(t) =& \frac{t^2}{4}\exp(-t) + \int_0^t \frac{(t-\tau)^2}{4}\exp{(\tau - t)}\varphi(\tau)d\tau, \\
\varphi(t) =& \sqrt{t} + 0.1 - t\sin t + \int_0^t \frac{\sin t\varphi^2(\tau)}{(\sqrt{\tau}+0.1)^2}d\tau, \\
\varphi(t) =& \frac{1}{3}\sin(3\varphi(t)) + \frac{t}{t^3-1}\int_0^t \frac{\exp(-t\tau + \tau^2\varphi^5(\tau))}{1+\varphi^2(\tau)}d\tau, \\
\varphi(t) =& \cos(t\varphi(t)) + \int_0^t \left(\ln(\tau t^2\varphi^2(\tau)) - \frac{\tan^{-1}(\tau + 2t^2 + \varphi^2(\tau))}{5 - \sqrt{|\sin(\varphi(\tau))|}}\right)d\tau,
\end{aligned}$$

the hypothesis of Theorem 3 are satisfied.

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